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A Problem on Arcs Without Bypasses in Tournaments

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An arc $\overrightarrow{xy}$ in a tournament is bad if there exists no path of length two from x to y. Formulas are found for the number of tournaments T_n whose bad arcs determine a spanning cycle or path.

1. INTRODUCTION

A *tournament* T_n consists of a set of *nodes* $1, 2, \ldots, n$ such that each pair of distinct nodes i and j is joined by exactly one of the *arcs* $\overrightarrow{ij}$ or $\overrightarrow{ji}$. If the arc $\overrightarrow{ij}$ is in T_n we say that i *beats* j or j *loses* to i and write $i \to j$; if each node in a subset A beats each node in a subset B we write $A \to B$. A sequence of nodes $\{p_1, p_2, \ldots, p_m\}$ in which $p_i \to p_{i+1}$ for $1 \leqslant i < m$ determines a *path* or *cycle* according as $p_1 \neq p_m$ or $p_1 = p_m$; we assume the nodes are distinct except that p_1 and p_m may be the same. A *spanning* path or cycle of T_n is one that contains every node of T_n. As a general reference on tournaments see [3].

An arc $\overrightarrow{xy}$ in a tournament T_n has a *bypass* (of length two) if there exists a node z in T_n such that $x \to z$ and $z \to y$. Alspach, Reid, and Roselle [1] have established certain results pertaining to bypasses of various lengths in asymmetric digraphs. Let us say that an arc $\overrightarrow{xy}$ in T_n is *good* if it has at least one bypass (of length two); otherwise it is *bad*. Clearly, no two arcs of the type $\overrightarrow{xu}$ and $\overrightarrow{xv}$ or $\overrightarrow{rx}$ and $\overrightarrow{sx}$ can both be bad; it follows, therefore, that the subgraph determined by the bad arcs of any tournament T_n consists of a (possibly empty) collection of disjoint paths and cycles. Our object here is to enumerate the tournaments T_n whose bad arcs determine a spanning cycle or path.

2. Tournaments Whose Bad Arcs Determine a Spanning Cycle

A tournament T_n is *transitive* if its nodes can be labeled in such a way that $i \to j$ if and only if $1 \leqslant i < j \leqslant n$. We shall use the following lemma in proving our main results.

Lemma. *Suppose the tournament T_n contains the arcs $\overrightarrow{12}, \overrightarrow{23}, \ldots, \overrightarrow{n-1, n}$ and that these arcs are bad; if $1 \to n$ then T_n is transitive.*

Proof. The result is certainly true when $n = 2$ so let us suppose that $n \geqslant 3$. If $i \to n$ for some node i where $1 \leqslant i \leqslant n-2$, then $i+1 \to n$ also since the arc $\overrightarrow{i, i+1}$ is bad; but $1 \to n$, by hypothesis, so it follows that $i \to n$ for all i such that $1 \leqslant i \leqslant n-1$. Furthermore, $1 \to n-1$ since $1 \to n$ and the arc $\overrightarrow{n-1, n}$ is bad; hence, we may assume that the subtournament $T_{n-1} = T_n - \{n\}$ is transitive. But then T_n is transitive also, by definition, and the result follows by induction.

Let $f(n)$ denote the number of tournaments T_n containing the arcs $\overrightarrow{12}, \overrightarrow{23}, \ldots, \overrightarrow{n-1, n}, \overrightarrow{n1}$ in which all these arcs are bad. (In what follows the labels of the nodes should be reduced modulo n when necessary; thus, for example, $n+1$ and 1 denote the same node.)

Theorem 1. *If $n \geqslant 3$, then $f(n) = 2^{n-1} - n$.*

Proof. If T_n is a tournament in which each arc $\overrightarrow{j, j+1}$ is bad, then if $j \to i$, where $i \neq j+1$, it must also be that $j+1 \to i$. This implies that for each node i in such a tournament there is a unique node β_i such that $i \to \{i+1, \ldots, \beta_i\}$ and $\{\beta_i + 1, \ldots, i-1\} \to i$. It follows from the lemma that the subtournaments determined by the nodes $\{i, i+1, \ldots, \beta_i\}$ and $\{\beta_i + 1, \ldots, i\}$ are transitive and that the orientations of the arcs in these subtournaments are consistent with the ordering indicated.

In particular, the subtournaments determined by $\{1, 2, \ldots, \beta_1\}$ and $\{\beta_1 + 1, \ldots, 1\}$ are transitive and this fact alone determines the orientation of all arcs of T_n except those joining nodes i and j where $2 \leqslant i \leqslant \beta_1$ and $\beta_1 + 1 \leqslant j \leqslant n$. This and the earlier observations imply that if $k = \beta_1$ then the numbers $\beta_1, \beta_2, \ldots, \beta_k$ determine the orientation of all arcs of T_n. It is not difficult to see that

$$2 \leqslant k \leqslant n-1, \tag{1}$$

and

$$i+1 \leqslant \beta_i \leqslant n \tag{2}$$

for $2 \leqslant i \leqslant k$; furthermore, since $i + 1 \to i + 2$ and the subtournament $\{i, i + 1,..., \beta_i\}$ is transitive, it follows that

$$\beta_i \leqslant \beta_{i+1} \tag{3}$$

for $1 \leqslant i \leqslant k - 1$. It cannot happen that

$$k = \beta_1 = \beta_2 = \cdots = \beta_k$$

because then the left inequality in (2) would not be satisfied when $i = k$. Thus every tournament T_n of the required type determines uniquely an integer k satisfying (1) and a set $B_k = \{\beta_1 ,..., \beta_k\}$ of k integers such that

$$k = \beta_1 \leqslant \beta_2 \leqslant \cdots \leqslant \beta_k \leqslant n \tag{4}$$

where we exclude the possibility that $k = \beta_1 = \cdots = \beta_k$.

Conversely, let k be any integer satisfying (1) and let $B_k = \{\beta_1 ,..., \beta_k\}$ be any set of k integers satisfying (4). If

$$k = \beta_1 \leqslant \cdots \leqslant \beta_i \leqslant i$$

for some $i \leqslant k$, then $i = k$ and $k = \beta_1 = \cdots = \beta_k$; thus if we exclude this last possibility it follows that inequalities (2) and (3) are satisfied and it is easy to see that there exists a tournament T_n of the required type associated with these parameters.

It follows, therefore, that $f(n)$ equals the number of solutions to inequalities (1) and (4), excluding the solutions $\beta_1 = \cdots = \beta_k$. The total number of sets B_k that satisfy (4), for a fixed value of k satisfying (1), is the number of selections of size $k - 1$ that can be chosen from the $n - k + 1$ numbers $k, k + 1,..., n$ where repetitions are permitted; the number of such selections (see [2, p. 36]) is

$$\binom{(n - k + 1) + (k - 1) - 1}{k - 1} = \binom{n - 1}{k - 1}.$$

If we exclude the nonadmissible selections we find that

$$f(n) = \sum_{k=2}^{n-1} \left\{\binom{n - 1}{k - 1} - 1\right\} = 2^{n-1} - n,$$

as required.

We remark that if $f(n, \alpha)$ denotes the number of these tournaments in which node 1 beats α nodes that are not beaten by node n, then it can be shown by induction that $f(n, 0) = 2^{n-2} - (n - 1)$ and that $f(n, \alpha) = 2^{n-2-\alpha}$ for $1 \leqslant \alpha \leqslant n - 2$.

3. Tournaments Whose Bad Arcs Determine a Spanning Path

Let $g(n)$ denote the number of tournaments T_n containing the arcs $\overrightarrow{12}, \overrightarrow{23},\ldots, \overrightarrow{n-1, n}$ in which all these arcs are bad; notice that we make no assumptions about the arc joining nodes 1 and n.

THEOREM 2. *If $n \geqslant 2$, then $g(n) = (1/n)\binom{2n-2}{n-1}$.*

Proof. Let T_n be a tournament in which each arc $\overrightarrow{j, j+1}$ is bad for $1 \leqslant j \leqslant n-1$; then if $j \to i$, where $1 \leqslant j \leqslant n-1$ and $i \neq j+1$, then $j+1 \to i$ also. This implies that for each node i, where $1 \leqslant i \leqslant n-1$, there exists a unique node β_i, where

$$i + 1 \leqslant \beta_i \leqslant n, \tag{5}$$

such that $i \to \{i+1,\ldots, \beta_i\}$ and, if $\beta_i < n$, then $\{\beta_i + 1,\ldots, n\} \to i$; thus if $1 \leqslant i < j \leqslant n$, then $i \to j$ if $j \leqslant \beta_i$ and $j \to i$ if $j > \beta_i$. The subtournament determined by the nodes $\{i, i+1,\ldots, \beta_i\}$ is transitive, by the lemma; this and the fact that $i+1 \to i+2$ imply that

$$\beta_i \leqslant \beta_{i+1} \tag{6}$$

for $1 \leqslant i \leqslant n-2$.

Conversely, if $B = \{\beta_1,\ldots, \beta_{n-1}\}$ denotes any set of $n-1$ integers that satisfy inequalities (5) and (6), it is easy to see that there exists a tournament T_n of the required type associated with these parameters. Hence, $g(n)$ equals the number of such sets B, or equivalently, the number of sets $A = \{\alpha_1, \alpha_2,\ldots, \alpha_{n-1}\}$ of $n-1$ integers such that

$$1 \leqslant \alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{n-1} \leqslant n-1$$

and

$$\alpha_i \geqslant i$$

for $1 \leqslant i \leqslant n-1$. It is well known and easy to show (see, e.g., [2, Chap. 3]) that there are $(1/n)\binom{2n-2}{n-1}$ such sets A. This suffices to complete the proof of the theorem.

Let $h(n)$ denote the number of tournaments T_n containing the arcs $\overrightarrow{12}, \overrightarrow{23},\ldots, \overrightarrow{n-1, n}$ in which these and only these arcs are bad. Theorems 1 and 2 imply the following result.

THEOREM 3. *If $n \geqslant 2$, then $h(n) = (1/n)\binom{2n-2}{n-1} - 2^{n-1} + n$.*

TABLE I

n	2	3	4	5	6	7	8
$f(n)$	0	1	4	11	26	57	120
$h(n)$	1	1	1	3	16	75	309

The first few values of $f(n)$ and $h(n)$ are listed in Table I. We remark that the total number of tournaments T_n with n labeled nodes whose bad arcs determine a spanning cycle or path is $(n-1)!\,f(n)$ or $n!\,h(n)$, respectively. It is easy to show (see [1] or [3, p. 32]) that the fraction of tournaments T_n with any bad arcs tends to zero rapidly as n tends to infinity. The problem of deriving a reasonable expression for the number of tournaments T_n with an arbitrary number of bad arcs seems very difficult in general.

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